

JOURNAL OF COMBINATORIAL THEORY, Series A **39**, 1–18 (1985)

Equilateral Convex Pentagons Which Tile the Plane

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Received April 23, 1983

It is shown that an equilateral convex pentagon tiles the plane if and only if it has two angles adding to 180° or it is the unique equilateral convex pentagon with angles A, B, C, D, E satisfying $A + 2B = 360^\circ$, $C + 2E = 360^\circ$, $A + C + 2D = 360^\circ$ ($A \cong 70.88^\circ$, $B \cong 144.56^\circ$, $C \cong 89.26^\circ$, $D \cong 99.93^\circ$, $E \cong 135.37^\circ$). © 1985 Academic Press, Inc.

Although the area of mathematical tilings has been of interest for a long time there is still much to be discovered. We do not even know which convex polygons tile the plane. Furthermore, for those polygons which do tile, new tilings are being found. It is known that all triangles and quadrilaterals tile the plane and those convex hexagons which do tile the plane have been classified. It is also known that no convex n -gon with $n \geq 7$ tiles.

In this paper we consider the problem of finding all equilateral convex pentagons which tile the plane. The upshot of our study is the following:

THEOREM. *An equilateral convex pentagon tiles the plane if and only if it has two angles adding to 180° , or it is the unique equilateral convex pentagon X with angles A, B, C, D, E satisfying $A + 2B = 360^\circ$, $C + 2E = 360^\circ$, $A + C + 2D = 360^\circ$ ($A \cong 70.88^\circ$, $B \cong 144.56^\circ$, $C \cong 89.26^\circ$, $D \cong 99.93^\circ$, $E \cong 135.37^\circ$).*

Thus the list of equilateral convex pentagons which tile, to be found in Schattschneider's paper [2], is complete. We also note that, with this theorem, the only convex polygons whose ability to tile is still in question are the nonequilateral convex pentagons.

It should be remarked that in obtaining this result we make no assumptions regarding periodicity of any tiling. (Yet it is a fact that every equilateral convex pentagon which tiles does so in a periodic manner.)

Our method of proof is interesting if only for the fact that it works only for the problem at hand—it could not, for instance, handle the problems of

finding all convex pentagons or all equilateral convex hexagons which tile. In various places in the proof computer calculations are used.

With no further ado, let us begin.

1. INITIAL REDUCTION

We can suppose that any tiling of an equilateral convex pentagon is edge-to-edge. For if not, it has one or more "fault-lines." It is easy to see that such fault-lines are necessarily parallel to one another, and that there are at most a countable infinity of them. So the tiling can be slipped along the fault-lines to become edge-to-edge (see Fig. 1.).

Thus we need concern ourselves only with the ways in which the angles match up at the vertices of the tiling, in other words, relations between the angles A, B, C, D, E of the pentagon of the form

$$m_A A + m_B B + m_C C + m_D D + m_E E = 360^\circ,$$

where m_A, \dots, m_E are nonnegative integers. First we show that the set of possible relations is *finite*. In an equilateral-convex pentagon, each angle is

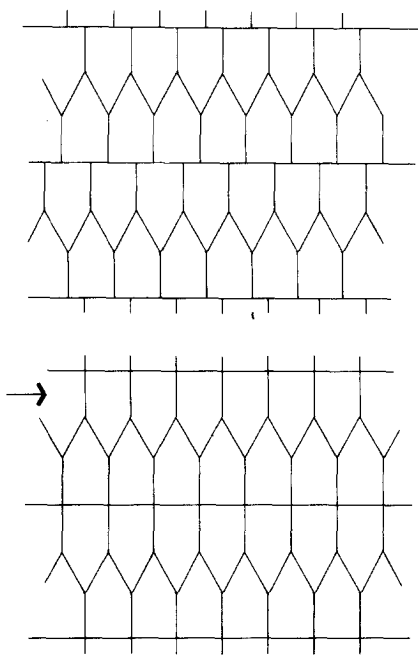


FIG. 1. Any tiling of an equilateral convex pentagon can be made edge-to-edge.

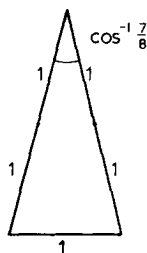


FIG. 2. In an equilateral convex pentagon, each angle is greater than $\cos^{-1}(\frac{7}{8})$.

greater than $\cos^{-1}(\frac{7}{8})$ (see Fig. 2), since if any angle were less than or equal to this, the polygon would fail to be a convex pentagon. So we have

$$A, B, C, D, E > \cos^{-1}(\frac{7}{8}) > 28^\circ$$

and since $m_A A + m_B B + \cdots + m_E E = 360^\circ$, we have

$$m_A + m_B + \cdots + m_E \leq 12.$$

Further, since A, B, C, D, E are all less than 180° ,

$$m_A + \cdots + m_E \geq 3.$$

Thus there are a finite (if large) set of relations that might be satisfied by some equilateral convex pentagon. We proceed to show how the above list may be whittled down.

There is a lot of duplication; if, for example, a pentagon satisfies $A + 2B = 360^\circ$, it also satisfies, after suitable relettering, any one of ten different relations (assuming, as we shall from now on, that the angles of the pentagon are A, B, C, D, E in that order around the pentagon.) We remove such trivial duplication by adopting, without loss of generality, the following conventions. We shall suppose, in everything that follows, that B is the largest angle of the pentagon, that is, $B \geq A, C, D, E$, and, further, that of the two angles adjacent to B , namely A and C , A is the smaller, that is, $A \leq C$.

With the above conventions, we have

LEMMA 1. $A \leq C \leq D \leq E \leq B$.

Proof. Suppose B is fixed, and A decreases from its value when equal to C (see Fig. 3(a)) when D is equal to E . As A decreases, E increases, D decreases and C increases, until E becomes equal to B and C becomes equal to D (see Fig. 3(b)). Thus we have

$$A \leq C \leq \lim C = \lim D \leq D \leq E \leq B,$$

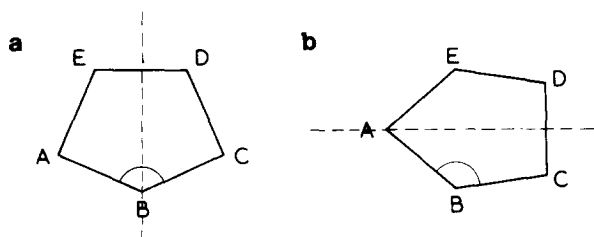


FIG. 3. (a) The greatest values of A occurs when $A = C$, $D = E$; (b) the least value of A occurs when $B = E$, $C = D$.

or

$$A \leq C \leq D \leq E \leq B. \quad \blacksquare$$

Lemma 1 will prove useful later in reducing the list of relations.

2. THE GEOMETRY OF AN EQUILATERAL CONVEX PENTAGON

In order to proceed we need to make a careful study of the geometry of the equilateral convex pentagon. Indeed, we prove

LEMMA 2. $108^\circ \leq B < 180^\circ$,

$$180^\circ - \frac{1}{2}B - \sin^{-1}(\sin(\frac{1}{2}B) - \frac{1}{2}) \geq A \geq 180^\circ - B + 2 \sin^{-1}(1/4 \sin(\frac{1}{2}B))$$

$$D = \cos^{-1}(\cos A + \cos B - \cos(A + B) - \frac{1}{2})$$

$$C = 270^\circ - B - \frac{1}{2}D + \theta$$

$$E = 270^\circ - A - \frac{1}{2}D - \theta,$$

where

$$\theta = \tan^{-1}((\sin A - \sin B)/(1 - \cos A - \cos B))$$

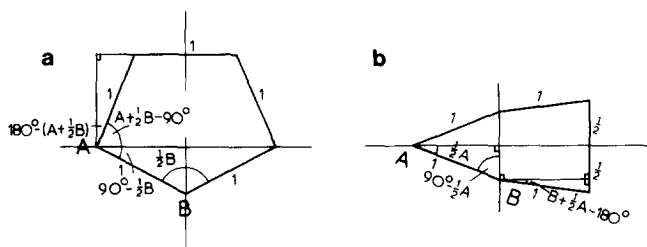


FIG. 4. (a) The greatest value of A is given by $\sin(\frac{1}{2}B) - \sin(180^\circ - (A + \frac{1}{2}B)) = \frac{1}{2}$; (b) the smallest value of A is given by $\sin(\frac{1}{2}A) + \sin(B + \frac{1}{2}A - 180^\circ) = \frac{1}{2}$.

Proof. The greatest value of A occurs when $A = C$ (see Fig. 4(a)), and then

$$\sin(\tfrac{1}{2}B) - \sin(180^\circ - (A + \tfrac{1}{2}B)) = \tfrac{1}{2}$$

or

$$A = 180^\circ - \tfrac{1}{2}B - \sin^{-1}(\sin(\tfrac{1}{2}B) - \tfrac{1}{2}).$$

The smallest value of A occurs when $B = E$ (see Fig. 4(b)), and then

$$\sin(\tfrac{1}{2}A) + \sin(B + \tfrac{1}{2}A - 180^\circ) = \tfrac{1}{2}$$

or

$$2 \sin(\tfrac{1}{2}A + \tfrac{1}{2}B - 90^\circ) \cos(90^\circ - \tfrac{1}{2}B) = \tfrac{1}{2}$$

or

$$\begin{aligned} 2 \sin(\tfrac{1}{2}A + \tfrac{1}{2}B - 90^\circ) \sin(\tfrac{1}{2}B) &= \tfrac{1}{2}, \\ A &= 180^\circ - B + 2 \sin^{-1}(1/4 \sin(\tfrac{1}{2}B)). \end{aligned}$$

In order to find D in terms of A and B , we calculate the length of the diagonal CE in two different ways. From Fig. 5(a) we see that

$$CE^2 = (2 \sin \tfrac{1}{2}D)^2$$

while from Fig. 5(b) we have

$$CE^2 = (1 - \cos A - \cos B)^2 + (\sin A - \sin B)^2.$$

Equating these expressions yields

$$\cos D = \cos A + \cos B - \cos(A + B) - \tfrac{1}{2}.$$

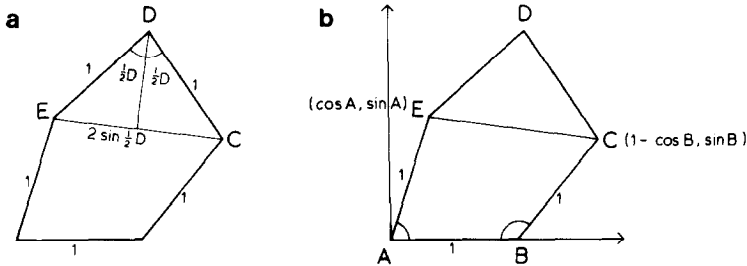


FIG. 5. (a) In an arbitrary equilateral convex pentagon the length of the diagonal CE is given by $CE = 2 \sin \tfrac{1}{2}D$; (b) an alternative expression for the length of the diagonal CE is given by $(CE)^2 = (1 - \cos A - \cos B)^2 + (\sin A - \sin B)^2$.

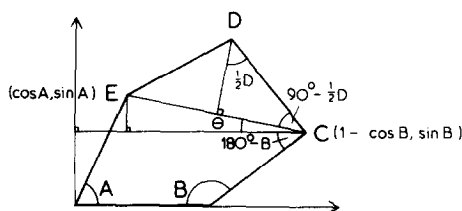


FIG. 6. This shows that $C = 270^\circ - B - \frac{1}{2}D + \theta$, and hence that $E = 270^\circ - A - \frac{1}{2}D - \theta$, where $\tan \theta = (\sin A - \sin B)/(1 - \cos A - \cos B)$.

Now from Fig. 6. it is easy to see that

$$C = (180^\circ - B) + (90^\circ - \frac{1}{2}D) + \theta = 270^\circ - B - \frac{1}{2}D + \theta$$

and

$$E = (180^\circ - A) + (90^\circ - \frac{1}{2}D) - \theta = 270^\circ - A - \frac{1}{2}D - \theta,$$

where

$$\tan \theta = (\sin A - \sin B)/(1 - \cos A - \cos B). \quad \blacksquare$$

In particular it follows from Lemma 2 that in an equilateral convex pentagon the angles C , D , and E are uniquely determined by the angles A and B , so we can identify the equilateral convex pentagon with a point in the AB plane. The region \mathcal{P} in the AB plane which results from this identification is shown in Fig. 7. We indicate certain polygons on the boundary

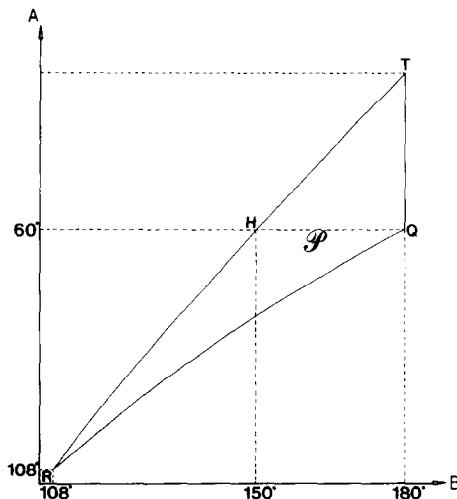


FIG. 7. Shows the region \mathcal{P} in the AB plane which corresponds to the set of all equilateral convex pentagons. The part of the boundary in the line $B = 180^\circ$ between T and Q consists of points representing quadrilaterals, not convex pentagons.

of the region. These are: R , the regular pentagon ($A = B = C = D = E = 108^\circ$); H , the house pentagon ($A = 60^\circ$, $B = E = 150^\circ$, $C = D = 90^\circ$); T , the isosceles triangle of Fig. 2 ($A = \cos^{-1}(\frac{7}{8})$, $B = E = 180^\circ$, $C = D = \cos^{-1}(\frac{1}{4})$); and Q , the quadrilateral which is half a regular hexagon ($A = C = 60^\circ$, $B = 180^\circ$, $D = E = 150^\circ$). Note that the part of the boundary of the region lying along the line $B = 180^\circ$ joining T and Q consists of (points representing) quadrilaterals, not convex pentagons.

3. FURTHER REDUCTION

Now $\cos^{-1}(\frac{7}{8}) (\cong 28.96^\circ) < A \leq 108^\circ$ and $108^\circ \leq B < 180$ in \mathcal{P} . Also $60^\circ < C \leq 108^\circ$, $\cos^{-1}(\frac{1}{4}) (\cong 75.52^\circ) < D < 120^\circ$, and $108^\circ \leq E < 180^\circ$ for all points $(A, B) \in \mathcal{P}$. (Note that these constraints are determined by values at Q , R , and T). These constraints allow just 220 solutions to the equation

$$m_A A + m_B B + m_C C + m_D D + m_E E = 360^\circ.$$

In order to determine which of our 220 relations are "good" in the sense that they are actually satisfied by some equilateral convex pentagon, we proceed as follows. For each set (m_A, \dots, m_E) we consider the function of A, B defined over the region \mathcal{P} by $m_A A + \dots + m_E E$, and which we will denote simply by $m_A m_B \dots m_E$. (Thus, for example, 20010 denotes $2A + D$.) The relation $m_A A + \dots + m_E E = 360^\circ$ is good if and only if the function $m_A m_B \dots m_E$ intersects the level set 360° for some $(A, B) \in \mathcal{P}$. However, rather than test each of the 220 functions in this way, we cut down the required work as follows: Define a partial order on the set of functions by writing

$$m_A \dots m_E < m'_A \dots m'_E$$

if $m_A A + \dots + m_E E \leq m'_A A + \dots + m'_E E$ simply by virtue of the fact that $A \leq C \leq D \leq E \leq B$. If $m_A m_B m_C m_D m_E < m'_A m'_B m'_C m'_D m'_E$ and $m'_A A + \dots + m'_E E < 360^\circ$ then $m_A A + \dots + m_E E < 360^\circ$. Hence we may discard such an $m_A \dots m_E$. Similarly if $m_A m_B m_C m_D m_E > m'_A m'_B m'_C m'_D m'_E$ and $m'_A A + \dots + m'_E E > 360^\circ$ then we may discard $m_A m_B m_C m_D m_E$.

LEMMA 3.

$$\begin{array}{ll} B + C + D < 360^\circ & A + C + D + E > 360^\circ \\ 2D + E < 360^\circ & 2A + 2C + E > 360^\circ \\ 2A + 2E > 360^\circ & 4C + D > 360^\circ \\ A + B + 2C > 360^\circ & 4A + C + E > 360^\circ \end{array}$$

$$2A + 4C > 360^\circ$$

$$7A + E > 360^\circ$$

$$5A + 3C > 360^\circ$$

$$8A + 2C > 360^\circ$$

$$10A + C > 360^\circ$$

Proof. Figure 8 shows regions of the AB plane and demonstrates that the thirteen associated equalities do not meet \mathcal{P} . ■

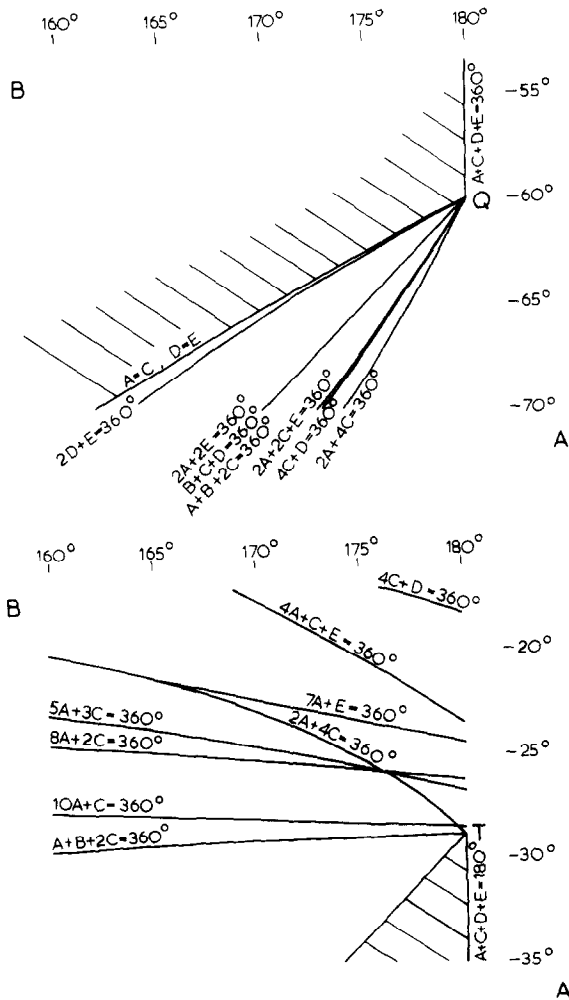


FIG. 8. Shows that certain angle relations are not satisfied by any equilateral convex pentagon.

The following 107 relations may be eliminated by use of the partial order $<$.

$< \underline{01110}$	$> \underline{20201}$	$> \underline{20400}$
00111	00230	00600
01200	11120	10320
00201	10121	10410
11010	11210	10500
10011	10211	20040
11100	11300	20130
10101	10301	20220
21000	22010	21300
20001	21011	20301
	20012	20310
$< \underline{00021}$	21020	20500
00030	20021	30220
	22100	30310
$> \underline{20002}$	21101	30400
22000	20102	40400
21001	21110	
32000	20111	$> \underline{70001}$
31001	21200	61010
30002		60011
	$> \underline{00410}$	61100
$> \underline{11200}$	00320	60101
01030	01400	71000
01120	00401	81000
02200		80001
01201	$> \underline{40101}$	
00202	31020	$> \underline{50300}$
01210	30021	50120
01300	31110	50210
13000	30111	60300
11002	31200	
11020	30201	
12100	42000	
	41001	
$> \underline{10111}$	40002	
00031	41010	
02116	40011	
01111	41100	
00112	41110	
00121	40111	
00211	41200	
12001	40201	
10003	51010	
12010	50011	
11011	51100	
10012	50101	
10021		
11101		
10102		
11110		

This leaves 100 relations. Each of these is satisfied by pentagons lying on a curve crossing the region \mathcal{P} (see Fig. 9).

LEMMA 4. *There are precisely 100 relations satisfied by some equilateral convex pentagon, namely:*

- | | | |
|------------------------------|-------------------------------|-------------------------------|
| 1. $A + 2E = 360^\circ$ | 36. $C + 3D = 360^\circ$ | 71. $3A + 3D = 360^\circ$ |
| 2. $A + B + E = 360^\circ$ | 37. $4D = 360^\circ$ | 72. $7A = 360^\circ$ |
| 3. $A + 2B = 360^\circ$ | 38. $5A = 360^\circ$ | 73. $6A + C = 360^\circ$ |
| 4. $C + 2E = 360^\circ$ | 39. $4A + C = 360^\circ$ | 74. $6A + D = 360^\circ$ |
| 5. $B + C + E = 360^\circ$ | 40. $4A + D = 360^\circ$ | 75. $6A + E = 360^\circ$ |
| 6. $2B + C = 360^\circ$ | 41. $4A + E = 360^\circ$ | 76. $6A + B = 360^\circ$ |
| 7. $B + 2D = 360^\circ$ | 42. $4A + B = 360^\circ$ | 77. $5A + 2C = 360^\circ$ |
| 8. $D + 2E = 360^\circ$ | 43. $3A + 2C = 360^\circ$ | 78. $5A + C + D = 360^\circ$ |
| 9. $B + D + E = 360^\circ$ | 44. $3A + C + D = 360^\circ$ | 79. $5A + 2D = 360^\circ$ |
| 10. $2B + D = 360^\circ$ | 45. $3A + C + E = 360^\circ$ | 80. $4A + 3C = 360^\circ$ |
| 11. $3E = 360^\circ$ | 46. $3A + B + C = 360^\circ$ | 81. $4A + 2C + D = 360^\circ$ |
| 12. $B + 2E = 360^\circ$ | 47. $3A + 2D = 360^\circ$ | 82. $4A + C + 2D = 360^\circ$ |
| 13. $2B + E = 360^\circ$ | 48. $3A + D + E = 360^\circ$ | 83. $4A + 3D = 360^\circ$ |
| 14. $3B = 360^\circ$ | 49. $3A + B + D = 360^\circ$ | 84. $8A = 360^\circ$ |
| 15. $4A = 360^\circ$ | 50. $2A + 3C = 360^\circ$ | 85. $7A + C = 360^\circ$ |
| 16. $3A + C = 360^\circ$ | 51. $2A + 2C + D = 360^\circ$ | 86. $7A + D = 360^\circ$ |
| 17. $3A + D = 360^\circ$ | 52. $2A + C + 2D = 360^\circ$ | 87. $6A + 2C = 360^\circ$ |
| 18. $3A + E = 360^\circ$ | 53. $2A + 3D = 360^\circ$ | 88. $6A + C + D = 360^\circ$ |
| 19. $3A + B = 360^\circ$ | 54. $A + 4C = 360^\circ$ | 89. $6A + 2D = 360^\circ$ |
| 20. $2A + 2C = 360^\circ$ | 55. $A + 3C + D = 360^\circ$ | 90. $9A = 360^\circ$ |
| 21. $2A + C + D = 360^\circ$ | 56. $A + 2C + 2D = 360^\circ$ | 91. $8A + C = 360^\circ$ |
| 22. $2A + C + E = 360^\circ$ | 57. $A + C + 3D = 360^\circ$ | 92. $8A + D = 360^\circ$ |
| 23. $2A + B + C = 360^\circ$ | 58. $A + 4D = 360^\circ$ | 93. $7A + 2C = 360^\circ$ |
| 24. $2A + 2D = 360^\circ$ | 59. $5C = 360^\circ$ | 94. $7A + C + D = 360^\circ$ |
| 25. $2A + D + E = 360^\circ$ | 60. $6A = 360^\circ$ | 95. $7A + 2D = 360^\circ$ |
| 26. $2A + B + D = 360^\circ$ | 61. $5A + C = 360^\circ$ | 96. $10A = 360^\circ$ |
| 27. $A + 3C = 360^\circ$ | 62. $5A + D = 360^\circ$ | 97. $9A + C = 360^\circ$ |
| 28. $A + 2C + D = 360^\circ$ | 63. $5A + E = 360^\circ$ | 98. $9A + D = 360^\circ$ |
| 29. $A + 2C + E = 360^\circ$ | 64. $5A + B = 360^\circ$ | 99. $11A = 360^\circ$ |
| 30. $A + C + 2D = 360^\circ$ | 65. $4A + 2C = 360^\circ$ | 100. $12A = 360^\circ$ |
| 31. $A + 3D = 360^\circ$ | 66. $4A + C + D = 360^\circ$ | |
| 32. $4C = 360^\circ$ | 67. $4A + 2D = 360^\circ$ | |
| 33. $3C + D = 360^\circ$ | 68. $3A + 3C = 360^\circ$ | |
| 34. $3C + E = 360^\circ$ | 69. $3A + 2C + D = 360^\circ$ | |
| 35. $2C + 2D = 360^\circ$ | 70. $3A + C + 2D = 360^\circ$ | |

4. PROOF OF MAIN THEOREM

We next observe that if an equilateral convex pentagon tiles the plane, it simultaneously satisfies at least *two* of the 100 relations. For, every angle of the tiling pentagon is involved in some such relation, yet no one relation

involves all five angles. So we must consider intersections of the 100 relations (see Fig. 9). Six of the relations coincide.

LEMMA 5. *The following relations are equivalent:*

- (A) 2. $A + B + E = 360^\circ$
- (B) 23. $2A + B + C = 360^\circ$
- (C) 25. $2A + D + E = 360^\circ$
- (D) 35. $2C + 2D = 360^\circ$
- (E) 44. $3A + C + D = 360^\circ$
- (F) 60. $6A = 360^\circ$.

Proof. If $A = 60^\circ$ then the pentagon is an equilateral triangle joined along one edge to a rhombus and it is clear that all 6 relations hold. Clearly $(A) \Leftrightarrow (D)$ since $A + B + C + D + E = 540^\circ$.

Now

$$A > 60^\circ \Rightarrow C + D > 180^\circ \Rightarrow 3A + C + D > 360^\circ$$

and

$$A < 60^\circ \Rightarrow C + D < 180^\circ \Rightarrow 3A + C + D < 360^\circ.$$

Hence $(A) \Leftrightarrow (D) \Leftrightarrow (E) \Leftrightarrow (F)$ and $(F) \Rightarrow (B)$, $(F) \Rightarrow (C)$.

We now show that $(B) \Rightarrow (F)$: Assume that $2A + B + C = 360^\circ$. Let $A = (0, 0)$ and $B(1, 0)$ then

$$D = (\cos A, \sin A)$$

$$C = (1 - \cos B, \sin B)$$

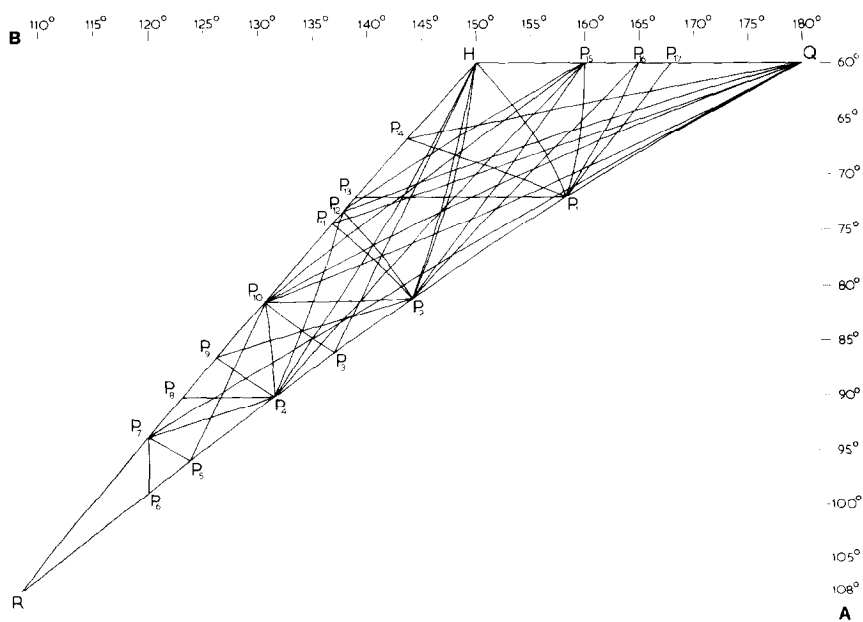
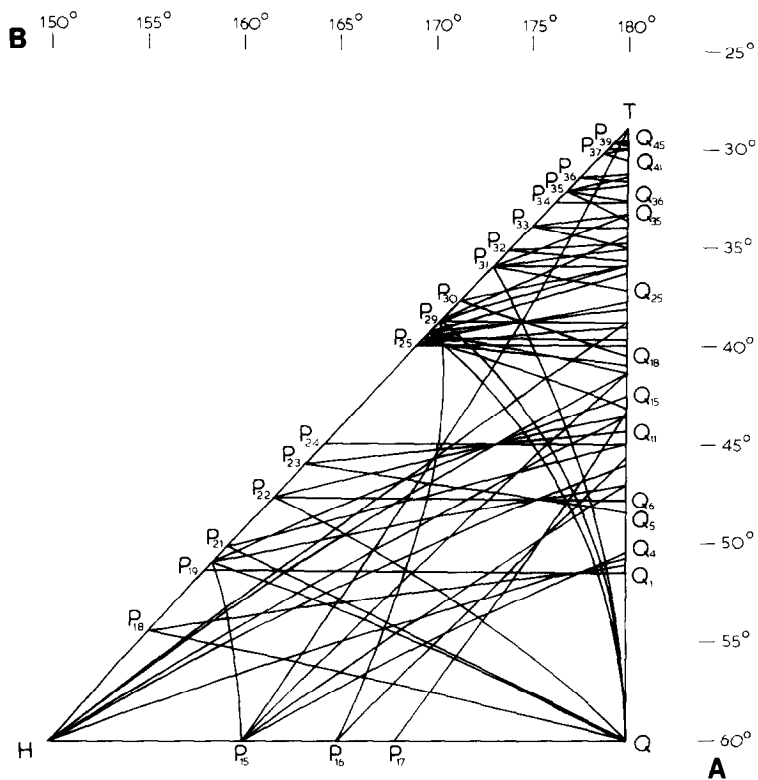
$$E = (1 - \cos B - \cos(B + C - 180^\circ), \sin B + \sin(B + C - 180^\circ)).$$

Now $(DE)^2 = 1$. Hence

$$(\cos A - 1 + \cos B - \cos(B + C))^2 + (\sin A - \sin B + \sin(B + C))^2 = 1.$$

Also $B + C = 360^\circ - 2A$, so

$$\begin{aligned} & 1 + \cos^2 A + \cos^2 B + \cos^2 2A - 2 \cos A - 2 \cos B + 2 \cos 2A \\ & + 2 \cos A \cos B - 2 \cos A \cos 2A - 2 \cos B \cos 2A + \sin^2 A \\ & + \sin^2 B + \sin^2 2A - 2 \sin A \sin B - 2 \sin A \sin 2A \\ & + 2 \sin B \sin 2A = 1 \end{aligned}$$



or

$$4 - 2 \cos A - 2 \cos B + 2 \cos 2A + 2 \cos(A + B) \\ - 2 \cos(2A - A) - 2 \cos(B + 2A) = 1$$

or

$$4 - 4 \cos A + 2 \cos 2A = 1 + 2 \cos B + 2 \cos(2A + B) - 2 \cos(A + B) \\ = 1 + 4 \cos(A + B) \cos A - 2 \cos(A + B).$$

Hence

$$3 - 4 \cos A + 2 \cos 2A = 2 \cos(A + B)(2 \cos A - 1)$$

or

$$1 - 4 \cos A + 4 \cos^2 A = 2 \cos(A + B)(2 \cos A - 1).$$

Hence $\cos A = \frac{1}{2}$ or $\cos A - \cos(A + B) = \frac{1}{2}$, which is not possible in \mathcal{P} except at R , and here $2A + B + C \neq 360^\circ$. So $\cos A = \frac{1}{2}$ and $A = 60^\circ$. To show $(C) \Rightarrow (F)$ we argue as above with E replacing B . ■

FIG. 9. Shows the 100 relations satisfied by some equilateral pentagon:

Curve	Endpoints	Curve	Endpoints	Curve	Endpoints	Curve	Endpoints
1	$H \quad Q_{22}$	26	$H \quad Q_{12}$	51	$Q \quad P_{20}$	76	$P_{37} \quad Q_{42}$
2	$Q \quad H$	27	$P_4 \quad P_{12}$	52	$P_{20} \quad Q_7$	77	$P_{27} \quad Q_{14}$
3	$P_3 \quad H$	28	$P_2 \quad P_{12}$	53	$P_{20} \quad Q_{15}$	78	$P_{27} \quad Q_{20}$
4	$P_{10} \quad Q_7$	29	$P_2 \quad T$	54	$P_1 \quad P_{29}$	79	$P_{27} \quad Q_{24}$
5	$Q \quad P_{10}$	30	$Q \quad P_{12}$	55	$Q \quad P_{29}$	80	$P_{35} \quad Q_{33}$
6	$P_3 \quad P_{10}$	31	$P_{12} \quad Q_4$	56	$P_{29} \quad Q_{22}$	81	$P_{35} \quad Q_{36}$
7	$P_4 \quad Q_{16}$	32	$P_4 \quad H$	57	$P_{29} \quad Q_{27}$	82	$P_{35} \quad Q_{37}$
8	$Q \quad P_{10}$	33	$P_2 \quad H$	58	$P_{29} \quad Q_{31}$	83	$P_{35} \quad Q_{39}$
9	$P_4 \quad P_{10}$	34	$P_2 \quad Q_9$	59	$P_1 \quad Q_{13}$	84	$P_{24} \quad Q_{10}$
10	$P_5 \quad P_{10}$	35	$Q \quad H$	60	$Q \quad H$	85	$P_{26} \quad Q_{17}$
11	$Q \quad P_7$	36	$H \quad Q_8$	61	$Q \quad P_{18}$	86	$P_{26} \quad Q_{21}$
12	$P_4 \quad P_7$	37	$H \quad Q_{16}$	62	$P_{18} \quad Q_2$	87	$P_{33} \quad Q_{29}$
13	$P_5 \quad P_7$	38	$P_1 \quad P_{13}$	63	$P_{30} \quad Q_{18}$	88	$P_{33} \quad Q_{32}$
14	$P_6 \quad P_7$	39	$P_1 \quad P_{14}$	64	$P_{30} \quad Q_{27}$	89	$P_{33} \quad Q_{34}$
15	$P_4 \quad P_8$	40	$Q \quad P_{14}$	65	$Q \quad P_{22}$	90	$P_{25} \quad Q_{19}$
16	$P_4 \quad P_9$	41	$Q \quad P_{21}$	66	$P_{22} \quad Q_6$	91	$P_{32} \quad Q_{28}$
17	$P_2 \quad P_9$	42	$P_{21} \quad Q_{10}$	67	$P_{22} \quad Q_{12}$	92	$P_{32} \quad Q_{30}$
18	$P_2 \quad P_{11}$	43	$P_1 \quad H$	68	$Q \quad P_{28}$	93	$P_{39} \quad Q_{43}$
19	$Q \quad P_{11}$	44	$Q \quad H$	69	$P_{28} \quad Q_{16}$	94	$P_{39} \quad Q_{44}$
20	$P_4 \quad P_{10}$	45	$Q \quad P_{31}$	70	$P_{28} \quad Q_{23}$	95	$P_{39} \quad Q_{45}$
21	$P_2 \quad P_{10}$	46	$P_{31} \quad Q_{29}$	71	$P_{28} \quad Q_{26}$	96	$P_{31} \quad Q_{27}$
22	$P_2 \quad H$	47	$H \quad Q_3$	72	$P_{19} \quad Q_1$	97	$P_{36} \quad Q_{38}$
23	$Q \quad H$	48	$P_{31} \quad Q_{25}$	73	$P_{23} \quad Q_5$	98	$P_{36} \quad Q_{40}$
24	$Q \quad P_{10}$	49	$P_{31} \quad Q_{34}$	74	$P_{23} \quad Q_{11}$	99	$P_{34} \quad Q_{35}$
25	$Q \quad H$	50	$P_1 \quad P_{20}$	75	$P_{37} \quad Q_{41}$	100	$P_{38} \quad Q_{42}$

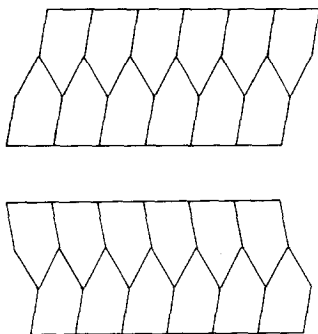


FIG. 10. Type 1 equilateral convex pentagons tile in a variety of ways.

Any equilateral convex pentagon which satisfies these 6 relations is said to be of Type 1.

The two relations

$$5. \quad B + C + E = 360^\circ$$

$$24. \quad 2A + 2D = 360^\circ$$

coincide, and any equilateral convex pentagon which satisfies these is said to be of Type 2(a).

The two relations

$$9. \quad B + D + E = 360^\circ$$

$$20. \quad 2A + 2C = 360^\circ$$

coincide, and any equilateral convex pentagon which satisfies these is said to be of Type 2(b). Together, Types 2(a) and 2(b) constitute Type 2.

Type 1 consists of all equilateral convex pentagons with two adjacent angles adding to 180° , while Type 2 consists of all equilateral convex pentagons with two nonadjacent angles adding to 180° . All these pentagons tile the plane (see Figs. 10, 11).

To find all other equilateral convex pentagons which tile the plane, we must consider all those which simultaneously satisfy at least two of the remaining 90 relations.

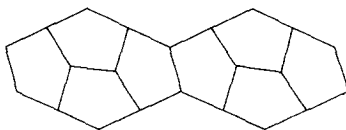


FIG. 11. Type 2 equilateral convex pentagons tile by forming hexagons with opposite pairs of sides parallel, which tile in strips.

We can shorten this task as follows: Any tiling pentagon satisfies at least one of the 14 relations involving B , namely:

- 3. $A + 2B = 360^\circ$
- 6. $2B + C = 360^\circ$
- 7. $B + 2D = 360^\circ$
- 10. $2B + D = 360^\circ$
- 12. $B + 2E = 360^\circ$
- 13. $2B + E = 360^\circ$
- 14. $3B = 360^\circ$
- 19. $3A + B = 360^\circ$
- 26. $2A + B + D = 360^\circ$
- 42. $4A + B = 360^\circ$
- 46. $3A + B + C = 360^\circ$
- 49. $3A + B + D = 360^\circ$
- 64. $5A + B = 360^\circ$
- 76. $6A + B = 360^\circ$

Further, the relations involving B that it satisfies cannot all belong to the second half of this list, since in each of these $m_A > m_B$, and so, in the tiling there would be more angles A than B , an impossibility. Thus the pentagon satisfies at least two relations, including at least one of the seven relations in the first half of the above list. There are 54 such pentagons, satisfying the following sets of relations:

3, 6	7, 11	7, 62	10, 13
3, 17	7, 17	7, 54	10, 12
3, 11	7, 18	7, 65	10, 15
3, 7, 21	7, 28	7, 72	10, 11
3, 18	7, 8	7, 36	10, 16
3, 8, 28	7, 33	7, 52	10, 17
3, 19, 38	7, 38	7, 66	11, 12, 13, 14
3, 4, 30	7, 19	7, 73	
3, 31	7, 39	7, 26, 42	
3, 39	7, 30	7, 68	
3, 40	7, 43	7, 55	
	7, 40	7, 84	
6, 7	7, 61	7, 74	
6, 11, 32	7, 47	7, 45, 67	
6, 17	7, 51	7, 53	
6, 27	7, 41	7, 77	

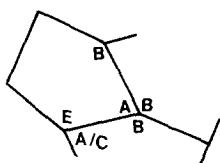


FIG. 12. Shows that a certain equilateral convex pentagon does not tile.

Of these 54, only three satisfy sets of relations involving all 5 angles:

$$3, 4, 30 \quad A + 2B = 360^\circ, \quad C + 2E = 360^\circ, \quad A + C + 2D = 360^\circ$$

$$3, 8, 28 \quad A + 2B = 360^\circ, \quad D + 2E = 360^\circ, \quad A + 2C + D = 360^\circ$$

$$7, 45, 67 \quad B + 2D = 360^\circ, \quad 3A + C + E = 360^\circ, \quad 4A + 2D = 360^\circ$$

Of these, the third does not tile since $m'_D > m_B$ in the only relation involving B . The second does not tile; for suppose it does, and consider any tiling of it near a vertex where $A + 2B = 360^\circ$. We see from Fig. 12 that E and A or E and C are forced together, but these combinations do not occur in any of the relations satisfied by that pentagon, yielding a contradiction.

This leaves only the first, which does tile the plane (see Fig. 13), and the theorem is proved.

5. FURTHER COMMENTS

A natural question to ask is: Given a tile which tiles the plane, can a description be given of all the distinct tilings of the plane with that tile?

For a general Type 1 or Type 2 tile (i.e., a tile which does not satisfy any additional relation) it is easy to answer this question. However for some tiles, and in particular the "versatile" $A = 60^\circ$, $B = 160^\circ$, $C = 80^\circ$, $D = 100^\circ$, $E = 140^\circ$, the question is hard to answer, since this tile satisfies no fewer than 11 angle relations. This tile and also the Type 2 tile with $A = 72^\circ$ yield tilings possessing only rotational symmetries illustrated in Figs. 14–16. One of these has appeared previously [1, p 159].

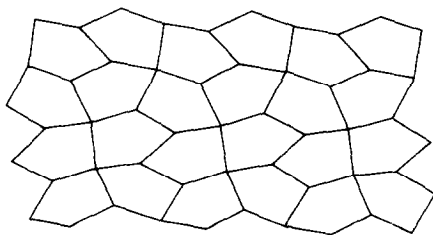


FIG. 13. The equilateral convex pentagon X with angles A , B , C , D , E satisfying $A + 2B = 360^\circ$, $C + 2E = 360^\circ$, $A + C + 2D = 360^\circ$ tiles the plane.

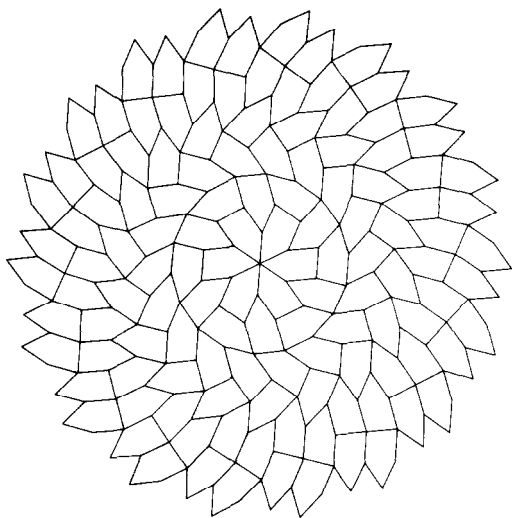


FIGURE 14

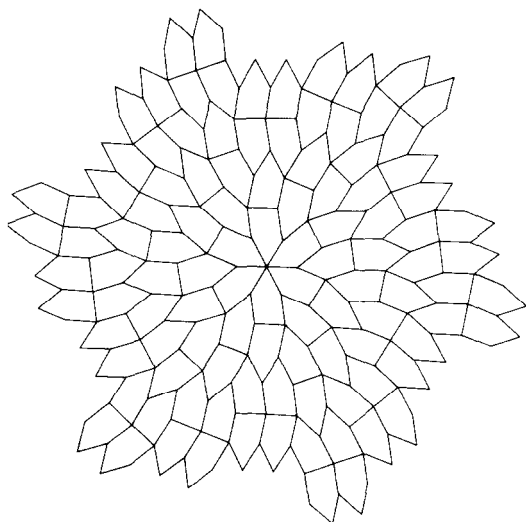


FIGURE 15

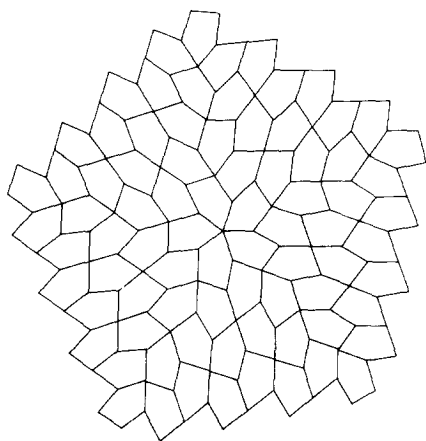


FIGURE 16

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